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Fractal random walks from a variational formalism for Tsallis entropies

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It is shown that random walks in which the set of visited points is a fractal emerge from a maximum entropy formalism applied to the generalized entropies introduced by Tsallis [J. Stat. Phys. **52**, 479 (1988)], upon suitable constraints. This connection between fractals and Tsallis entropies suggests that the generalized statistical mechanics derived from the latter could provide a natural frame for studying fractally structured systems.

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Entropy plays a central role in the foundations of equilibrium and nonequilibrium statistical mechanics. In the theory of discrete-time random walks, it is well known that the probability distribution of displacement per step $p(\mathbf{x})$ for a purely diffusive motion can be obtained from the requirement that the Boltzmann entropy

$$S[p(\mathbf{x})] = -k_B \int p(\mathbf{x}) \ln p(\mathbf{x})\, d\mathbf{x} \tag{1}$$

is extremal under variation of $p(\mathbf{x})$. This variational problem has to be solved under auxiliary conditions, namely, probability normalization,

$$\int p(\mathbf{x})\, d\mathbf{x} = 1, \tag{2}$$

and an additional constraint given by the finiteness of the mean square displacement per step,

$$\int x^2\, p(\mathbf{x})\, d\mathbf{x} = \sigma^2 d, \tag{3}$$

where $x = |\mathbf{x}|$ and d is the spatial dimension. Such a maximum entropy formalism [1] produces for the jump probability

$$p(\mathbf{x}) = A\, \exp(-x^2/2\sigma^2), \tag{4}$$

i.e., a Gaussian distribution, normalized by the constant A.

Pure diffusive motion is directly implied by the quadratic behavior of the characteristic function (Fourier transform) of $p(\mathbf{x})$ near the origin:

$$G(\mathbf{k}) \approx 1 - \frac{1}{2}\sigma^2 k^2 + \cdots, \tag{5}$$

with $k = |\mathbf{k}|$. The same behavior is obtained for alternative forms of the constraint (3). For instance, defining the mean jump length $\int x p(\mathbf{x}) d\mathbf{x} = \langle x \rangle d$ produces a Poissonian distribution, $p(\mathbf{x}) \propto \exp(-x/\langle x \rangle)$, for which $G(\mathbf{k}) \approx 1 - \langle x \rangle^2 k^2 + \cdots$.

This formalism fails, however, to describe random walks with more complex jump probabilities. Among them, the so-called Lévy flights [1,2] are determined by a jump probability whose characteristic function reads

$$G(\mathbf{k}) = \exp(-\alpha k^\gamma) \approx 1 - \alpha k^\gamma + \cdots, \tag{6}$$

with positive α and $0 < \gamma < 2$. It can be easily shown that the corresponding $p(\mathbf{x})$—called Lévy distribution—behaves for large x as

$$p(\mathbf{x}) \sim x^{-1-\gamma}. \tag{7}$$

For any $\gamma > 0$, this power-law asymptotic behavior characterizes *stable* distributions.

The absence of a characteristic length scale for a long-tailed distribution as in (7), indicates that the set of points visited by the walker is a self-similar structure, namely, a fractal. Its fractal dimension can be determined to equal γ [3]. As an illustration, Fig. 1 shows the first 10^4 points visited during a two-dimensional random walk with jump probability

$$p(\mathbf{x}) = B\,(1+x)^{-1-\gamma}, \tag{8}$$

where $\gamma = 1.6$ and B is a normalization constant.

In the traditional frame of Boltzmann-Gibbs statistics, the application of the maximum entropy formalism to these fractal random walks would require forcing the jump probability to satisfy rather artificial or unconventional constraints, with no connection with any of the main thermodynamical averages. Typically, besides normalization, these constraints would involve the specification of the average value of a complicated logarithmic function [4], instead of a simple constraint like the mean value defined in (3). As stressed by Montroll and Shlesinger, “the wonderful world of clusters and intermittencies and bursts that is associated with Lévy distributions would be hidden from us if we depended on a maximum entropy formalism that employed simple traditional auxiliary conditions” [1]. Note that the maximum entropy formalism can be modified both by considering unconventional constraints *and/or by varying the definition of entropy.*

As in the case of Lévy flights, Boltzmann-Gibbs statistics seems to be inappropriate in dealing with a class of physical systems which involve long-range interactions.

FIG. 1. The first 10^4 points visited by a random walker on the plane, with the jump probability density given in Eq. (8). The arrow indicates the initial position of the walker. The fractal dimension of this pattern is $\gamma = 1.6$.

In astrophysics, for instance, it provides an infinite mass for the polytropic model of stellar systems as well as similar inconsistencies for the three-dimensional gravitational N-body problem [5]. It is also unable to describe the scaling laws observed in the relevant fields of vortex systems [6]. Similar problems appear in the theory of black holes [7] and superstrings [8].

These problems—all of which are probably related to nonextensivity in the involved systems [9]—could be solved in the frame of the generalized statistics proposed by Tsallis [10]. This generalization consists in defining the entropy of a system whose ith microscopic state has probability p_i as

$$S_q[p_i] \equiv -k \, \frac{1-\sum_i p_i^q}{1-q}, \qquad q \in \mathbb{R} \tag{9}$$

where k is a conventional positive constant. For $q \to 1$ and $k = k_B$ the usual Boltzmann definition, $S = -k_B \sum_i p_i \ln p_i$, is recovered.

Under conditions of probability normalization, $\sum_i p_i = 1$, and a generalized form of energy average,

$$\sum_i \epsilon_i p_i^q = E_q, \tag{10}$$

the maximization of S_q makes it possible to derive a generalized thermodynamics, which is formally equivalent to the traditional one up to the level of its Legendre-transformation structure [11]. The nonadditivity of Tsallis entropy stands for its connection with nonextensive systems [9,11].

It has been shown that the generalized statistical mechanics is consistent with suitable forms of the Ehrenfest theorem and Jaynes duality relations [12], von Neumann equation [13], fluctuation-dissipation theorem [14], Bogolyubov inequality [15], Langevin and Fokker-Planck equations [16], and Callen identity [17]. Tsallis entropy has also been applied to quantum statistics [18]. Up to this moment, the main success of this generalized statistics in the applications has been the solution that it was able to provide [19] for the divergent mass in the polytropic model quoted before.

In the spirit of the maximum entropy formalism, Tsallis entropy can be applied to random walks if suitably extended for a vectorial continuum variable $\mathbf{x}$ in a d-dimensional space. Let $p(\mathbf{x})$ be its associated probability distribution, which is required to satisfy the normalization condition (2). The corresponding extended form of Tsallis entropy reads

$$S_q[p(\mathbf{x})] = -k \, \frac{1-\int p(\mathbf{x})^q d\mathbf{x}}{1-q}. \tag{11}$$

The natural extension of the generalized constraint (10) in terms of x^2 is [cf. Eq. (3)]

$$\int x^2 p(\mathbf{x})^q d\mathbf{x} = \Omega_d \int x^{d-1} x^2 p(x)^q dx = \sigma_q^2 d, \tag{12}$$

where Ω_d is the total solid angle in the d-dimensional space.

Applying the variational principle to $S_q[p(\mathbf{x})]$ upon the constraints (2) and (12), the following form for the jump probability is obtained:

$$p(\mathbf{x}) = \left[\frac{kq}{\alpha(q-1)} + \frac{\beta q}{\alpha}\, x^2\right]^{1/(1-q)}, \tag{13}$$

where α and β are the variational Lagrange parameters. Note that this result—*which does not depend on the spatial dimension*—is analogous to the energy probability distribution obtained in the generalized thermodynamics [11].

The jump probability distribution in (13) has the same asymptotic functional form as in (7). Thus, with a convenient choice of the index q, the set of points visited by the walker will be a self-similar, fractal structure. Requiring that $p(\mathbf{x}) \sim x^{-1-\gamma}$ for large x implies

$$q = \frac{3+\gamma}{1+\gamma}. \tag{14}$$

This value of q fixes the generalized statistics compatible with the fractal of dimension γ. In order to have a well-defined self-similar structure, this fractal dimension must be lower than the spatial dimension, $\gamma < d$. Furthermore, the normalization constraint (2) imposes $\gamma > d-1$. These requirements determine that the value of q given in (14) will satisfy $(3+d)/(1+d) < q < (2+d)/d$. In particular, this implies

$$1 < q < 3. \tag{15}$$

Observe that the limit $q \to 1$ is obtained for $\gamma \to \infty$, as all the moments of $p(\mathbf{x})$ become finite and the distribution is no longer stable. In fact, in this limit $p(\mathbf{x})$ reduces to a Gaussian distribution.

It is important to stress that the convergence of the integral in the generalized constraint (12) *does not impose an additional condition on the values of* γ *and* q. Indeed, for $x \to \infty$ the integrand $x^{d-1}x^2 p(\mathbf{x})^q$ behaves precisely as $x^{d-1}p(\mathbf{x})$,

$$x^{d+1}p^q \sim x^{d-1}p \sim x^{d-2-\gamma}, \tag{16}$$

so that the normalization of $p(\mathbf{x})$ just ensures the finiteness of σ_q^2. In this sense, the constraint (12)—which had been proposed *ad hoc* in the generalization of thermodynamics from Tsallis entropy [10,11] to preserve its Legendre structure—turns out to be a natural one.

These results extend the maximum entropy formalism to a class of processes characterized by long-tailed distributions which originate patterns with no scale lengths. The main ingredient in this path from statistics to self-similar structures is the use of Tsallis generalized entropies. Even though a connection between generalized thermodynamics and fractals had been already conjectured [10,20], here such a relation is effectively revealed.

Considering the increasing interest in fractals since the pioneering work by Mandelbrot [21], it seems necessary to construct a statistical-mechanical frame for fractal structures. Our results determine a relation between the index q in Tsallis entropy and the fractal dimension of the self-similar patterns generated from it, as well as justify to some extent the type of constraints used in the variational calculations. This suggests that Tsallis statistics could provide the proper tool for that purpose.

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